

Scattering QCD amplitudes with massive fermions using recursive relations

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Abstract

We study the QCD scattering amplitudes for $q\bar{q} \rightarrow gg$ and $q\bar{q} \rightarrow ggg$, where q is a massive fermion. Recursive calculational technique based on the method of basis spinors is used.

Introduction

When evaluating a Feynman amplitude involving fermions, the amplitude is expressed as sum of terms which have the form

$$\mathcal{M}_{\lambda_p, \lambda_k}(p, s_p, k, s_k; Q) = \mathcal{M}_{\lambda_p, \lambda_k}([p], [k]; Q) = \bar{u}_{\lambda_p}(p, s_p) Q u_{\lambda_k}(k, s_k) , \quad (1)$$

where λ_p and λ_k are spin indices of the external fermions with four-momenta p, k and arbitrary polarization vectors s_p, s_k . The operator Q is a sum of products of Dirac γ -matrices.

The main aim of calculation is to transform (1) to explicitly scalar form (inner products of four-vectors and so on).

The main approach, which has gained popularity in the past decades, is to calculate Feynman amplitudes directly. Many different methods of calculating reaction amplitudes with fermions have been developed.

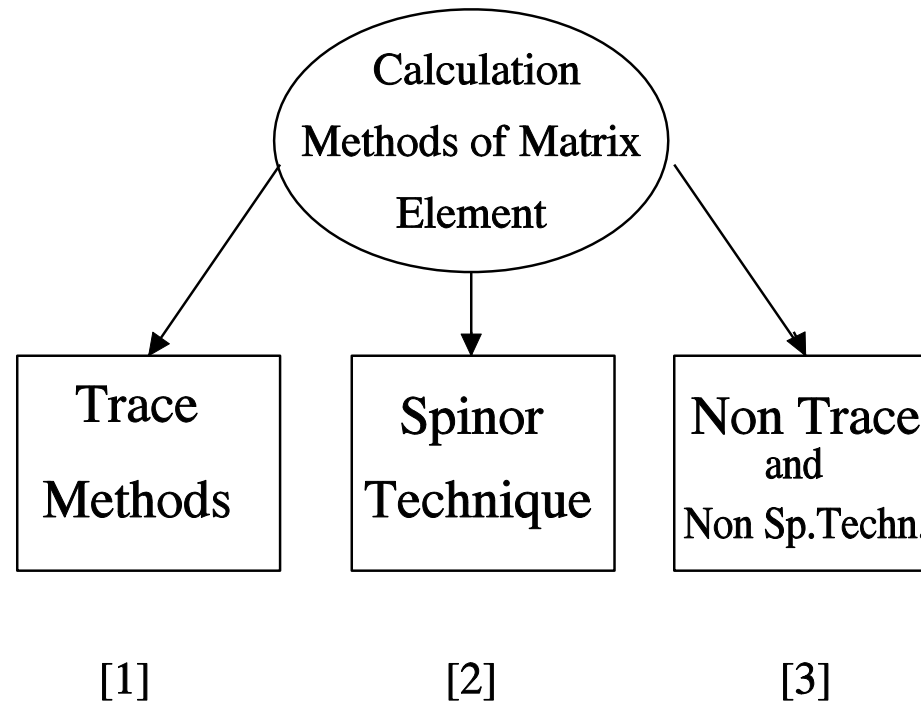


Figure 1: Diagram of the matrix elements calculation methods

In general, these methods of Feynman amplitudes calculation can be divided into three groups

[1]- Trace methods:

In this approach matrix element is reduced to trace calculations of γ -matrices products

A.A. Bogush, F.I. Fedorov (1962); A.A. Bogush(1964); E. Bellomo (1961); H.W. Fearing, R.R. Sillbar (1972); F.I. Fedorov(1980); J.D. Bjorken and M.C. Chen(1966); M. Caffo, E. Remiddi (1982); M.V.

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Galynski, S.M. Sikach, hep-ph/9910284; R. Vega and J. Wudka, (1996).

[2]- Spinor Technique:

In this method the matrix element is reduced to spinor products of Dirac spinors, i.e.

$\bar{u}_{\lambda_p}(p, s_p) u_{\lambda_k}(k, s_k)$. Spinor products are calculated through momentum components by means of traces.

F.A. Berends, P.H. Daverveldt, and R. Kleiss(1985); R. Kleiss, W.J. Stirling(1985); Zhan Xu, Da-Hua Zhang, and Lee Chang(1987); F.A. Berends and W.T. Giele(1987); F.A. Berends, P.H. Daverveldt, and R. Kleiss(1985); R. Kleiss(1987); S. Dittmaier(1999); V.V. Andreev (2000).

[3]- Nontrace and Non Spinor Techniques: In these methods algorithms of reduction of matrix element to scalar form differ from the above mentioned methods.

A. Ballestrero and E. Maina(1995); K. Hagiwara, D. Zeppenfeld (1986)

V.V. Andreev, NIM (2003); It should be noted that there are also methods of calculating cross sections without Feynman diagrams (see *F. Caravaglios, M. Moretti(1995); A. Kanaki and C.G. Papadopoulos(2000)*

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Isotropic tetrad

Let us introduce the orthonormal four-vector basis in Minkowski space which satisfies the relations:

$$l_0^\mu \cdot l_0^\nu - l_1^\mu \cdot l_1^\nu - l_2^\mu \cdot l_2^\nu - l_3^\mu \cdot l_3^\nu = g^{\mu\nu}, \quad (l_A \cdot l_B) = g_{AB}, \quad (2)$$

where g is the Lorentz metric tensor.

With the help of vectors l_A we can define lightlike vectors, which form the isotropic tetrad in Minkowski space

$$b_\rho = (l_0 + \rho l_3)/2, \quad n_\lambda = (\lambda l_1 + i l_2)/2, \quad (\rho, \lambda = \pm 1). \quad (3)$$

From Eqs. (2), (3) it follows that

$$(b_\rho \cdot b_{-\lambda}) = \frac{\delta_{\lambda,\rho}}{2}, \quad (n_\lambda \cdot n_{-\rho}) = \frac{\delta_{\lambda,\rho}}{2}, \quad (b_\rho \cdot n_\lambda) = 0, \quad (4)$$

$$g^{\mu\nu} = \sum_{\lambda=-1}^1 \left[\tilde{b}_\lambda^\mu \cdot b_{-\lambda}^\nu + \tilde{n}_\lambda^\mu \cdot n_{-\lambda}^\nu \right], \quad \tilde{b}_\lambda^\mu = 2 b_\lambda^\mu, \quad \tilde{n}_\lambda^\mu = 2 n_\lambda^\mu. \quad (5)$$

Massless basis spinors

With the help of isotropic tetrad

$$b_\rho = \frac{l_0 + \rho l_3}{2}, \quad n_\lambda = \frac{l_1 + i\lambda l_2}{2}$$

we define *basis spinors*

$$u_\lambda (b_{-1}) \text{ and } u_\lambda (b_1) :$$

$$\not{b}_{-1} u_\lambda (b_{-1}) = 0, \quad u_\lambda (b_1) \equiv \not{b}_1 u_{-\lambda} (b_{-1}), \quad (6)$$

$$\omega_\lambda u_\lambda (b_A) = u_\lambda (b_A), \quad (A = \pm 1) \quad (7)$$

with matrix $\omega_\lambda = 1/2 (1 + \lambda \gamma_5)$ and normalization condition

$$u_\lambda (b_A) \bar{u}_\lambda (b_A) = \omega_\lambda \not{b}_A. \quad (8)$$

Some properties of basis spinors

Phase condition.

If we introduce raising and lowering spin operators

$$\not{n}_\lambda u_{-\rho}(b_{-1}) = \delta_{\lambda,\rho} u_\lambda(b_{-1}) \quad (9)$$

we can fix the phases of the spinors $u_\lambda(b_{-1})$ and $u_\lambda(b_1)$.

The important property of basis spinors (6) is the

Completeness relation

$$\sum_{\lambda, A=-1}^1 u_\lambda(b_A) \bar{u}_{-\lambda}(b_{-A}) = \mathbf{I} \quad , \quad (10)$$

which follows from Eqs.(6),(9). Thus, the arbitrary bispinor can be decomposed in terms of basis spinors $u_\lambda(b_A)$.

Dirac spinors and basis spinors

Arbitrary Dirac spinor can be determined through the basis spinor (6) with the help of projection operators $\tau_\lambda (p, s_p) = u_{\lambda_p} (p, s_p) \bar{u}_{\lambda_p} (p, s_p)$.

The Dirac spinors $w_\lambda^A (p, s_p)$ for massive fermion and antifermion with four-momentum p ($p^2 = m_p^2$), arbitrary polarization vector s_p and spin number $\lambda = \pm 1$ can be obtained with the help of basis spinors by means of equation:

$$\begin{aligned} w_\lambda^A (p, s_p) &= \frac{(\not{p} + Am_p) (1 + \lambda \gamma_5 \not{s}_p)}{2\sqrt{(b_{-1} \cdot (p + m_p s_p))}} u_{-A \times \lambda} (b_{-1}) = \\ &= T_\lambda^A (p, s_p) u_{-A \times \lambda} (b_{-1}) . \end{aligned} \quad (11)$$

The notation $w_{\lambda_p}^A (p, s_p)$ stands for either $u_{\lambda_p} (p, s_p)$ (bispinor of fermion; $A = +1$) or $v_{\lambda_p} (p, s_p)$ (bispinor of antifermion; $A = -1$).

Bispinors $u_\lambda (p, s_p)$ and $v_\lambda (p, s_p)$ satisfy Dirac equations and spin conditions for massive

fermion and antifermion

$$\begin{aligned} \not{p} u_\lambda(p, s_p) &= m_p u_\lambda(p, s_p), & \not{p} v_\lambda(p, s_p) &= -m_p v_\lambda(p, s_p), \\ \gamma_5 \not{s}_p u_\lambda(p, s_p) &= \lambda u_\lambda(p, s_p), & \gamma_5 \not{s}_p v_\lambda(p, s_p) &= \lambda v_\lambda(p, s_p). \end{aligned} \quad (12)$$

We also found that Dirac spinors of fermions and antifermions are related by

$$v_\lambda(p, s_p) = -\lambda \gamma_5 u_{-\lambda}(p, s_p), \quad \bar{v}_\lambda(p, s_p) = \bar{u}_{-\lambda}(p, s_p) \lambda \gamma_5. \quad (13)$$

An arbitrary bispinor $u_\lambda(p)$ of momentum p ($p^2 = 0, (p \cdot b_{-1}) \neq 0$) and helicity λ is defined in terms of basis spinor as

$$u_\lambda(p) = \frac{\not{p}}{\sqrt{2} (p \cdot b_{-1})} u_{-\lambda}(b_{-1}). \quad (14)$$

Main equations of MBS

Spinor products of basis spinors are simple and similar to scalar products of isotropic tetrad vectors

$$\bar{u}_\lambda (b_C) u_\rho (b_A) = \delta_{\lambda,-\rho} \delta_{C,-A}, \quad C, A = \pm 1; \quad \lambda, \rho = \pm 1. \quad (15)$$

Using Eq. (5) Dirac matrix γ^μ can be represented in the form

$$\gamma^\mu = \sum_{\lambda=-1}^1 \left[\not{b}_{-\lambda} \tilde{b}_\lambda^\mu - \not{n}_{-\lambda} \tilde{n}_\lambda^\mu \right] \quad (16)$$

and with the help of Eqs.(7), (9) and (16) we can obtain that

$$\{\gamma^\mu\}_{\alpha\beta} \{u_\lambda (b_A)\}_\beta = \tilde{b}_A^\mu \{u_{-\lambda} (b_{-A})\}_\alpha - A \tilde{n}_{-A \times \lambda}^\mu \{u_{-\lambda} (b_A)\}_\alpha. \quad (17)$$

and

$$\gamma_5 u_\rho (b_A) = \rho u_\rho (b_A) . \quad (18)$$

Eqs. (15), (17)-(18) underlie the method of basis spinors (MBS).

By means of Eq.(17) we can determine that product of two γ -matrices can be represented as

$$\gamma^\mu \gamma^\nu u_\lambda (b_A) = Y_{A, \lambda}^{\mu, \nu} u_\lambda (b_A) - A X_{A, \lambda}^{\mu, \nu} u_\lambda (b_{-A}) , \quad (19)$$

where $X^{\mu, \nu}, Y^{\mu, \nu}$ are Lorentz tensors:

$$X_{A, \lambda}^{\mu, \nu} = \tilde{b}_A^\mu \cdot \tilde{n}_{-A \times \lambda}^\nu - \tilde{n}_{-A \times \lambda}^\mu \cdot \tilde{b}_A^\nu , \quad (20)$$

$$Y_{A, \lambda}^{\mu, \nu} = \tilde{b}_{-A}^\mu \cdot \tilde{b}_A^\nu + \tilde{n}_{A \times \lambda}^\mu \cdot \tilde{n}_{-A \times \lambda}^\nu . \quad (21)$$

The product $\mathcal{S}^n = \gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_n}$ can be written as

$$\begin{aligned} \mathcal{S}^n u_\lambda (b_A) &= \\ &= \mathcal{B}_{A, \lambda}^{\{\mu_1, \dots, \mu_n\}} u_{\lambda'_n} (b_{A'_n}) - A \mathcal{N}_{A, \lambda}^{\{\mu_1, \dots, \mu_n\}} u_{\lambda'_n} (b_{-A'_n}) , \end{aligned} \quad (22)$$

where

$$\lambda'_n = (-1)^n \lambda , \quad A'_n = (-1)^n A \quad (23)$$

and $\mathcal{B}_{A, \lambda}^{\{\mu_1, \dots, \mu_n\}}, \mathcal{N}_{A, \lambda}^{\{\mu_1, \dots, \mu_n\}}$ are some Lorentz tensors, which are related to isotropic tetrad vectors.

MBS

The **basic idea of Method of Basis Spinors** is to replace Dirac spinors in Eq.(1) by massless basis spinors $u_\lambda (b_{\pm 1})$ (Eq.(11)), and to use only three Eqs.

$$\bar{u}_\lambda (b_C) u_\rho (b_A) = \delta_{\lambda, -\rho} \delta_{C, -A} \quad , \quad \gamma_5 u_\rho (b_A) = \rho u_\rho (b_A)$$

$$\gamma^\mu u_\lambda (b_A) = \tilde{b}_A^\mu u_{-\lambda} (b_{-A}) - A \tilde{n}_{-A \times \lambda}^\mu u_{-\lambda} (b_A)$$

to calculate matrix element (1) in terms of scalar functions \mathcal{B}, \mathcal{N} . With the help of Eq.(11) matrix element (1) transforms to fermion “string” with massless basis spinors $u_\lambda (b_A)$ i.e.

$$\begin{aligned} \mathcal{M}_{\lambda_p, \lambda_k} (p, s_p, k, s_k; Q) &= \bar{u}_{\lambda_p} (p, s_p) Q u_{\lambda_k} (k, s_k) = \\ &= \bar{u}_{-\lambda_p} (b_{-1}) T_{\lambda_p} (p, s_p) Q T_{\lambda_k} (k, s_k) u_{-\lambda_k} (b_{-1}) = \\ &= \mathcal{M}_{-\lambda_p, -\lambda_k} (b_{-1}, b_{-1}; T_{\lambda_p} (p, s_p) Q T_{\lambda_k} (k, s_k)) \quad , \end{aligned} \quad (24)$$

where operator T_λ is determined by Eq.(11).

Basic matrix element

Let us consider an important type of matrix element (1), when $p = b_{-C}$ and $k = b_A$, i.e.

$$\mathcal{M}_{\sigma,-\rho}(b_C, b_{-A}; Q) \equiv \Gamma_{\sigma,\rho}^{C,A}[Q] = \bar{u}_\sigma(b_C) Q u_{-\rho}(b_{-A}) . \quad (25)$$

We call this type of matrix element as **basic matrix element**.

Note that matrix element (1) is a particular case of basic matrix element i.e.

$$\mathcal{M}_{\lambda_p,\lambda_k}(p, s_p, k, s_k; Q) = \Gamma_{\lambda_p,-\lambda_k}^{1,-1} [T_{\lambda_p}(p, s_p) Q T_{\lambda_k}(k, s_k)] . \quad (26)$$

With the help of completeness relation (10) we can obtain the recursive formula for $\Gamma_{\sigma,\rho}^{C,A}[Q_1 Q_2]$

$$\Gamma_{\sigma,\rho}^{C,A}[Q_1 Q_2] = \sum_{D,\lambda=-1}^1 \Gamma_{\sigma,\lambda}^{C,D}[Q_1] \Gamma_{\lambda,\rho}^{D,A}[Q_2] . \quad (27)$$

By means of MBS relations it is easy to calculate $\Gamma_{\sigma,\rho}^{C,A}$ in terms of the isotropic tetrad vectors.

For instance,

$$\Gamma_{\sigma, \rho}^{C, A} [\gamma^\mu] = \delta_{\sigma, -\rho} \left(\delta_{C, -A} \tilde{b}_A^\mu - A \delta_{C, A} \tilde{n}_{-A \times \rho}^\mu \right) , \quad (28)$$

$$\Gamma_{\sigma, \rho}^{C, A} [\gamma^\mu \gamma^\nu] = \delta_{\sigma, \rho} \left(\delta_{C, A} Y_{A, \rho}^{\mu, \nu} - A \delta_{C, -A} X_{A, \rho}^{\mu, \nu} \right) , \quad (29)$$

$$\begin{aligned} \Gamma_{\sigma, \rho}^{C, A} [\gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_n}] &= \Gamma_{\sigma, \rho}^{C, A} [\mathcal{S}^n] = \\ &= \delta_{\sigma, \rho'_n} \left(\delta_{C, A'_n} \mathcal{B}_{A, \rho}^{\{\mu_1, \dots, \mu_n\}} - A \delta_{C, -A'_n} \mathcal{N}_{A, \rho}^{\{\mu_1, \dots, \mu_n\}} \right) . \end{aligned} \quad (30)$$

With the help of the Eqs.(27) and (30) we obtain recursive relations for $\mathcal{B}_{A, \lambda}^{\{\mu_1, \dots, \mu_n\}}$ and $\mathcal{N}_{A, \lambda}^{\{\mu_1, \dots, \mu_n\}}$

$$\mathcal{B}_{A, \lambda}^{\{\mu_1, \dots, \mu_n\}} = \mathcal{B}_{A'_{n-k}, \lambda'_{n-k}}^{\{\mu_1, \dots, \mu_k\}} \mathcal{B}_{A, \lambda}^{\{\mu_{k+1}, \dots, \mu_n\}} + (-1)^{n-k+1} \mathcal{N}_{-A'_{n-k}, \lambda'_{n-k}}^{\{\mu_1, \dots, \mu_k\}} \mathcal{N}_{A, \lambda}^{\{\mu_{k+1}, \dots, \mu_n\}} , \quad (31)$$

$$\mathcal{N}_{A, \lambda}^{\{\mu_1, \dots, \mu_n\}} = \mathcal{B}_{-A'_{n-k}, \lambda'_{n-k}}^{\{\mu_1, \dots, \mu_k\}} \mathcal{N}_{A, \lambda}^{\{\mu_{k+1}, \dots, \mu_n\}} + (-1)^{n-k} \mathcal{N}_{A'_{n-k}, \lambda'_{n-k}}^{\{\mu_1, \dots, \mu_k\}} \mathcal{B}_{A, \lambda}^{\{\mu_{k+1}, \dots, \mu_n\}} . \quad (32)$$

The recursive Eqs.(31)-(32) allow us to convert scalar functions \mathcal{B}, \mathcal{N} into Lorentz tensors in terms of isotropic tetrad vector.

Building blocks of matrix elements

With the help of completeness relation (10) the amplitude (1) with $Q = Q_2 Q_1$ is expressed as combinations of the lower-order matrix elements

$$\mathcal{M}_{\lambda_p, \lambda_k}(p, k; Q_2 Q_1) = \sum_{\sigma, A=-1}^1 \mathcal{M}_{\lambda_p, \sigma}(p, b_A; Q_2) \mathcal{M}_{-\sigma, \lambda_k}(b_{-A}, k; Q_1) . \quad (33)$$

This insertion allows us to “cut” fermion chain (1) into pieces of fermion chains with basis spinors $u_\lambda(b_A)$ (see Fig.2)

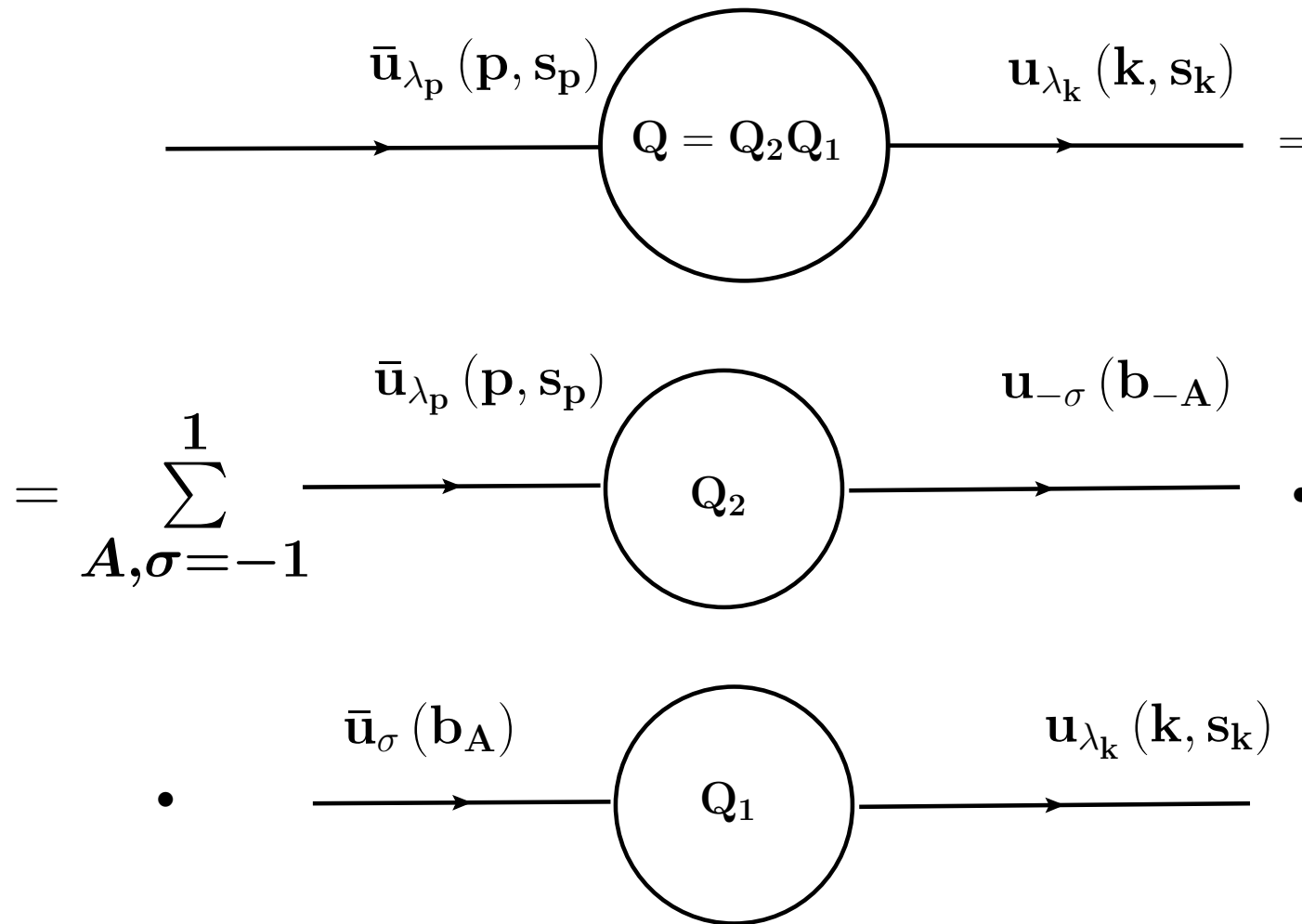


Figure 2: Diagram of the recursive relation (33)

Technique of “building” blocks. Applications.

As an example of how to use in practice MBS, let us consider the QCD scattering amplitudes for $q\bar{q} \rightarrow gg$, where q is a massive fermion with color index $i = 1, \dots, N_c$ and g is gluon with color index $A = 1, \dots, N_c^2 - 1$:

$$q_i(p_1, \lambda_1) + \bar{q}_j(p_2, \lambda_2) \rightarrow g_A(k_1, \sigma_1) + g_B(k_2, \sigma_2) . \quad (34)$$

In the past years, various new methods for efficient calculations in QCD have been introduced, motivated by the relation of QCD amplitudes to twistor string theory found in [E. Witten \(2004\) \[hep-th/0312171\]](#). In particular, these methods include the diagrammatic rules of CSW ([F. Cachazo, P. Svrcek, E. Witten\(2004\) \[hep-th/0403047\]](#)), where tree level QCD amplitudes are constructed from vertices that are off-shell continuations of maximal helicity violating (MHV) amplitudes, and recursive relations of BCFW ([R. Britto, F. Cachazo, B. Feng, and E. Witten \(2005\)\[hep-th/0501052\]](#)) that construct scattering amplitudes from on-shell amplitudes with external momenta shifted into the complex plane. The BCFW recursion relations have found numerous applications in tree level and one-loop calculations in QCD ([S.Weinzierl\(2007\)\[hep-ph/0707.3342\]](#)).

The main advantages of the BCFW and CSW constructions are in the simplification of analytical calculations as compared to more traditional off-shell recursive methods ([F. A. Berends and W. T. Giele \(1988\)](#)). All above mentioned methods are based on spinor techniques and use special Feynman rules.

We specify the kinematics in the CM system. Identifying the beam axes with the z direction and taking the xz plane as the scattering plane, the momenta read

$$p_1 = \frac{\sqrt{s}}{2}(1, 0, 0, \beta_q) , k_1 = \frac{\sqrt{s}}{2}(1, \sin \theta, 0, \cos \theta) ,$$

$$p_2 = \frac{\sqrt{s}}{2}(1, 0, 0, -\beta_q) , k_2 = \frac{\sqrt{s}}{2}(1, -\sin \theta, 0, -\cos \theta) , \quad (35)$$

$$\varepsilon_{\sigma_1}^\mu(k_1) = \frac{1}{\sqrt{2}}(0, \cos \theta, i \sigma_1, -\sin \theta) , \varepsilon_{\sigma_2}^\mu(k_2) = \frac{1}{\sqrt{2}}(0, \cos \theta, -i \sigma_2, -\sin \theta) , \quad (36)$$

The Mandelstam variables are defined as usual:

$$s = (p_1 + p_2)^2 , t = (p_1 - k_1)^2 , u = (p_1 - k_2)^2 ,$$

$$t_m = t - m_q^2 , u_m = u - m_q^2 , r^2 = tu - m_q^4 \quad (37)$$

$$\beta_q = \sqrt{1 - \frac{4m_q^2}{s}} , \gamma_q = \frac{2m_q}{\sqrt{s}} \quad (38)$$

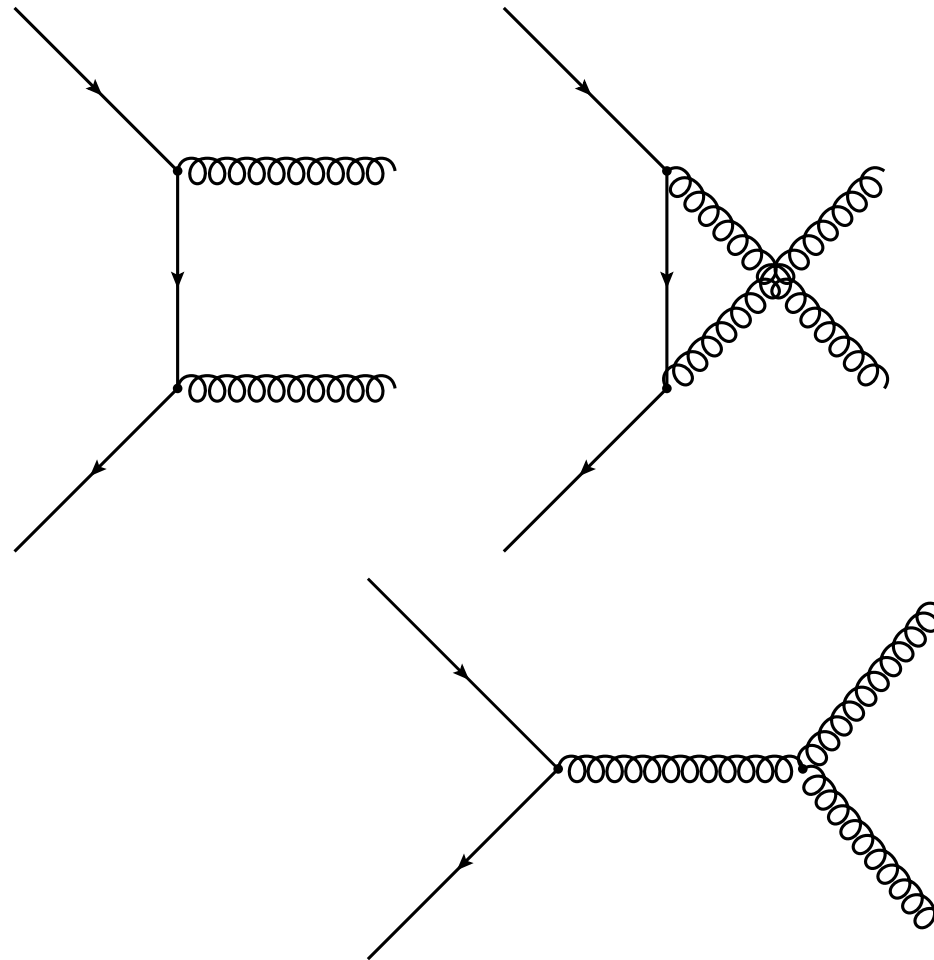


Figure 3: Diagram of $q(p_1, \lambda_1) + \bar{q}(p_2, \lambda_2) \rightarrow g(k_1, \sigma_1) + g(k_2, \sigma_2)$ in Born approximation

$$\begin{aligned} \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (1) &= -g_s^2 t_{ik}^b t_{kj}^a \times \\ &\times \bar{v}_{\lambda_2} (p_2, s_{p_2}) \not{\epsilon}_{\sigma_2} (k_2) \frac{\not{p}_1 - \not{k}_1 + m_q}{t - m_q^2} \not{\epsilon}_{\sigma_1} (k_1) u_{\lambda_1} (p_1, s_{p_1}) \end{aligned} \quad (39)$$

$$\begin{aligned} \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (2) &= -g_s^2 t_{ik}^a t_{kj}^b \times \\ &\times \bar{v}_{\lambda_2} (p_2, s_{p_2}) \not{\epsilon}_{\sigma_2} (k_2) \frac{\not{p}_1 - \not{k}_2 + m_q}{u - m_q^2} \not{\epsilon}_{\sigma_1} (k_1) u_{\lambda_1} (p_1, s_{p_1}) \end{aligned} \quad (40)$$

$$\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (3) = i \frac{g_s^2}{s} f^{bac} t_{ij}^c \bar{v}_{\lambda_2} (p_2, s_{p_2}) K u_{\lambda_1} (p_1, s_{p_1}) \quad (41)$$

$$K = (k_1 - k_2) (\varepsilon_{\sigma_2} (k_2) \cdot \varepsilon_{\sigma_1} (k_1)) + 2 [\varepsilon_{\sigma_2} (k_2) (k_2 \cdot \varepsilon_{\sigma_1} (k_1)) - \varepsilon_{\sigma_1} (k_1) (k_1 \cdot \varepsilon_{\sigma_2} (k_2))] \quad (42)$$

$$K = -\delta_{\sigma_2, \sigma_1} \sqrt{s} (0, \sin \theta, 0, \cos \theta) \quad (43)$$

Using completeness relation for basis spinors, the matrix elements $\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (1, 2, 3)$ of the

$q\bar{q} \rightarrow gg$ can be written as:

$$\begin{aligned} \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (1) &= -\frac{g_s^2}{t_m} t_{ik}^b t_{kj}^a \sum_{A, \rho=-1}^1 \mathcal{M}_{\lambda_2, -\rho} (p_2, b_{-A}; \mathbf{I}) \times \\ &\times [(-1) \mathcal{M}_{\rho, \lambda_1} (b_A, p_1; \not{\epsilon}_{\sigma_2} (k_2) \not{k}_1 \not{\epsilon}_{\sigma_1} (k_1)) + \\ &+ 2 (\varepsilon_{\sigma_1} (k_1) \cdot p_1) \mathcal{M}_{\rho, \lambda_1} (b_A, p_1; \not{\epsilon}_{\sigma_2} (k_2))] \end{aligned} \quad (44)$$

$$\begin{aligned} \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (2) &= -\frac{g_s^2}{u_m} t_{ik}^a t_{kj}^b \sum_{A, \rho=-1}^1 \mathcal{M}_{\lambda_2, -\rho} (p_2, b_{-A}; \mathbf{I}) \times \\ &\times [(-1) \mathcal{M}_{\rho, \lambda_1} (b_A, p_1; \not{\epsilon}_{\sigma_1} (k_1) \not{k}_2 \not{\epsilon}_{\sigma_2} (k_2)) + \\ &+ 2 (\varepsilon_{\sigma_2} (k_2) \cdot p_1) \mathcal{M}_{\rho, \lambda_1} (b_A, p_1; \not{\epsilon}_{\sigma_1} (k_1))] \end{aligned} \quad (45)$$

$$\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (3) = i \frac{g_s^2}{s} f^{bac} t_{ij}^c \sum_{A, \rho=-1}^1 \mathcal{M}_{\lambda_2, -\rho} (p_2, b_{-A}; \mathbf{I}) \mathcal{M}_{\rho, \lambda_1} (b_A, p_1; \mathbf{K}) \quad (46)$$

Therefore for $q\bar{q} \rightarrow gg$ we have three “building” blocks:

Block I

$$\mathcal{M}_{\rho, -\lambda_p}^{(B)}(b_A, p; \mathbf{I}) = u_\rho(b_A) w_{\lambda_p}^B(p,) \equiv s_{\rho, \lambda_p}^{A, B}(p) = \Gamma_{\rho, -\lambda_p}^{A, -1} [T_{\lambda_p}(p, s_p)] \quad (47)$$

consist of decomposition coefficients of a Dirac spinor (fermion or antifermion) into basis spinors $u_\lambda(b_A)$, $A, \lambda = \pm 1$.

Block II (Current-like)

$$\mathcal{M}_{\rho, \lambda_p}(b_C, p; \gamma^\mu) = \sum_{A, \rho=-1}^1 \Gamma_{\rho, \sigma}^{C, A}[\gamma^\mu] s_{\sigma, \lambda_p}^{A, 1}(p) \quad (48)$$

Block III

$$\mathcal{M}_{\rho, \lambda_p}(b_C, p; \gamma^\mu \gamma^\nu \gamma^\alpha) = \sum_{A, \rho=-1}^1 \Gamma_{\rho, \sigma}^{C, A}[\gamma^\mu \gamma^\nu \gamma^\alpha] s_{\sigma, \lambda_p}^{A, 1}(p) \quad (49)$$

Decomposition coefficients of fermion helicity states in CM frame can be easily calculated

Andreev(2004):

$$s_{\rho,\lambda}^{(A,1)}(p) = -\frac{\lambda \sqrt[4]{s}}{\sqrt{2}} \delta_{A\rho,-\lambda} \sqrt{1 - \lambda \rho \beta_q}, \quad \beta_q = \sqrt{1 - \frac{4m_q^2}{s}} \quad (50)$$

and using that

$$\Gamma_{\sigma,\rho}^{C,A} [\gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_n}] = \Gamma_{\sigma,\rho}^{C,A} [\mathcal{S}^n] = \delta_{\sigma,\rho'_n} \left(\delta_{C,A'_n} \mathcal{B}_{A,\rho}^{\{\mu_1,\dots,\mu_n\}} - A \delta_{C,-A'_n} \mathcal{N}_{A,\rho}^{\{\mu_1,\dots,\mu_n\}} \right) .$$

we have the universal block for any number of γ -matrices:

$$\begin{aligned} \widetilde{\mathcal{M}}_{\lambda_2,\lambda_1}(p_2,p_1;\mathcal{S}^n) &\equiv \widetilde{\mathcal{M}}_{\lambda_2,\lambda_1}(\mathcal{S}^n) = \\ &= (-1)^n \frac{\sqrt{s}}{2} \sum_{\sigma=-1}^1 \left(\delta_{\lambda_2,-\lambda'_1} \frac{2m_q}{\sqrt{s}} + (1 - \sigma \beta_q \lambda_1) \delta_{\lambda_2,\lambda'_1} \right) \times \\ &\times \left[\delta_{\lambda_2,-\lambda_1} \mathcal{N}_{\sigma \times \lambda_1, -\sigma}^{\{n\}} - \sigma \lambda_1 \delta_{\lambda_2,\lambda_1} \mathcal{B}_{\sigma \times \lambda_1, -\sigma}^{\{n\}} \right], \quad \lambda' = (-1)^n \lambda \quad . \end{aligned} \quad (51)$$

The recursive Eqs.(31)-(32) allow us to convert scalar functions \mathcal{B}, \mathcal{N} into Lorentz tensors in terms of isotropic tetrad vector with the help of Eq.(20) or Eq.(21).

Matrix element can be rewritten in terms of the block $\widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\mathcal{S}^n)$. For example,

$$\begin{aligned} \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1}(1) = & -\frac{g_s^2}{t_m} t_{ik}^b t_{kj}^a \left[\widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\not{\epsilon}_{\sigma_2}(k_2) \not{k}_1 \not{\epsilon}_{\sigma_1}(k_1)) + \right. \\ & \left. + 2(\varepsilon_{\sigma_1}(k_1) \cdot p_1) \widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\not{\epsilon}_{\sigma_2}(k_2)) \right] \end{aligned} \quad (52)$$

Using Eqs. (28),(29), (31)-(32) and (51), tree level amplitudes with two external gluons and arbitrary spin indices $\lambda_{1,2}$ and $\sigma_{1,2}$ (as contrasted to spinor technique) may be written in the form

$$\begin{aligned} \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1}(1) = & -\frac{g_s^2 t_{ik}^b t_{kj}^a s}{2 t_m} \left[\lambda_1 \delta_{\lambda_2, -\lambda_1} \sin \theta (\beta_q (\cos \theta + \lambda_1 \sigma_1 \delta_{\sigma_2, -\sigma_1}) - \delta_{\sigma_2, \sigma_1}) + \right. \\ & \left. + \delta_{\lambda_2, \lambda_1} \gamma_q (\beta_q \sin^2 \theta + (\cos \theta + \lambda_1 \sigma_1) \delta_{\sigma_2, \sigma_1}) \right] , \end{aligned} \quad (53)$$

$$\begin{aligned} \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1}(2) = & -\frac{g_s^2 t_{ik}^a t_{kj}^b s}{2 u_m} \left[\lambda_1 \delta_{\lambda_2, -\lambda_1} \sin \theta (\beta_q (\cos \theta + \lambda_1 \sigma_1 \delta_{\sigma_2, -\sigma_1}) + \delta_{\sigma_2, \sigma_1}) \right. \\ & \left. + \delta_{\lambda_2, \lambda_1} \gamma_q (\beta_q \sin^2 \theta + (\lambda_1 \sigma_1 - \cos \theta) \delta_{\sigma_2, \sigma_1}) \right] , \end{aligned} \quad (54)$$

$$\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1}(3) = i g_s^2 f^{bac} t_{ij}^c (\lambda_1 \delta_{\lambda_2, -\lambda_1} \sin \theta - \gamma_q \delta_{\lambda_2, \lambda_1} \cos \theta) \delta_{\sigma_2, \sigma_1} , \quad (55)$$

where

$$\cos \theta = \frac{t - u}{\beta_q s}, \quad \sin \theta = \frac{2r}{\beta_q s}, \quad r^2 = tu - m_q^4. \quad (56)$$

Sum of $\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1}$ can be rewritten as

$$\begin{aligned} \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} &= \frac{1}{2} \{t^b, t^a\} (\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (1) + \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (2)) + \\ &+ \frac{1}{2} [t^b, t^a] (\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (1) - \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (2) + 2\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1} (3)) \end{aligned} \quad (57)$$

When squared, scattering amplitudes (57) give the well-known result (see for example [Borodulin, V. I. and Rogalev, R. N. and Slabospitsky, S. R. IHEP95 CORE -Compendium of Relations](#)), namely ($N_c = 3$)

$$|\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_2, \sigma_1}|^2 = 16g_s^4 \chi_0 \left(\frac{8}{3} - 6\chi_1 \right) \quad (58)$$

with additional functions

$$\chi_0 = \frac{t_m}{u_m} + \frac{u_m}{t_m} + \frac{4m_q^2 s}{t_m u_m} \left(1 - \frac{m_q^2 s}{t_m u_m} \right), \quad \chi_1 = \frac{t_m u_m}{s^2}. \quad (59)$$

Let us consider some matrix elements of the process

$$q_i(p_1, \lambda_1) + \bar{q}_j(p_2, \lambda_2) \rightarrow g_A(k_1, \sigma_1) + g_B(k_2, \sigma_2) + g_C(k_3, \sigma_3) . \quad (60)$$

There are 16 diagrams. Some of them are shown in Fig. 4

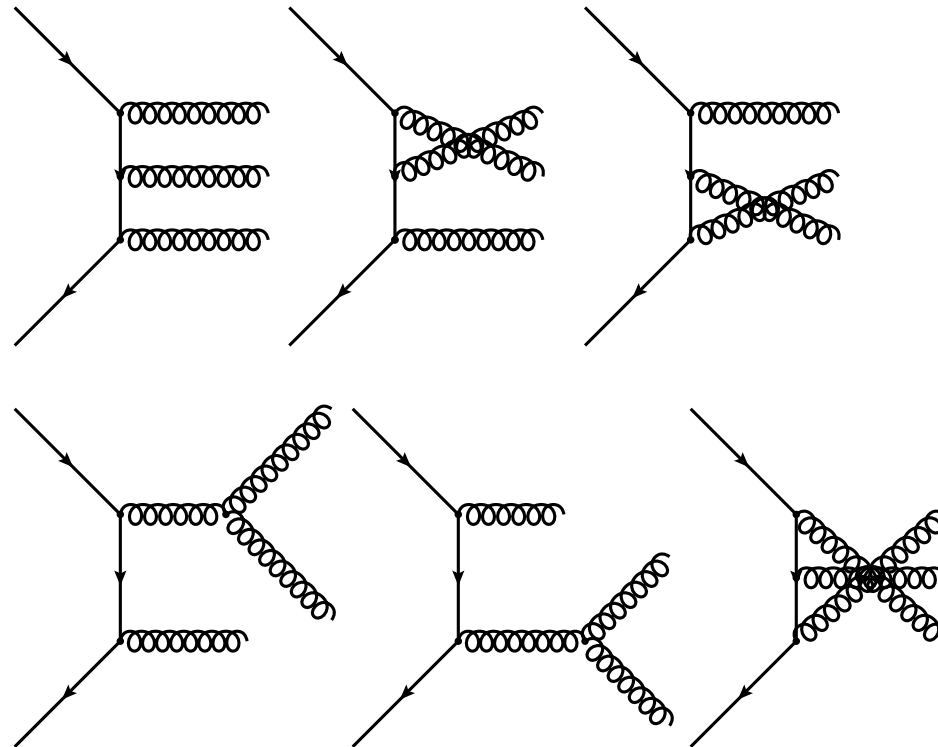


Figure 4: Some tree-level diagrams of $q(p_1, \lambda_1) + \bar{q}(p_2, \lambda_2) \rightarrow g(k_1, \sigma_1) + g(k_2, \sigma_2) + g(k_3, \sigma_3)$

For reaction $q(p_1, \lambda_1) + \bar{q}(p_2, \lambda_2) \rightarrow g(k_1, \sigma_1) + g(k_2, \sigma_2) + g(k_3, \sigma_3)$ the main“building” blocks are shown in Fig. 5. There is only one additional function $\Gamma[\gamma^\alpha \hat{Q}_2 \gamma^\nu \hat{Q}_1 \gamma_\mu]$ as compared with the reaction $q(p_1, \lambda_1) + \bar{q}(p_2, \lambda_2) \rightarrow g(k_1, \sigma_1) + g(k_2, \sigma_2)$. This function is reduced by lower-order blocks with the help of recursive relation for the basic matrix element $\Gamma[Q]$.

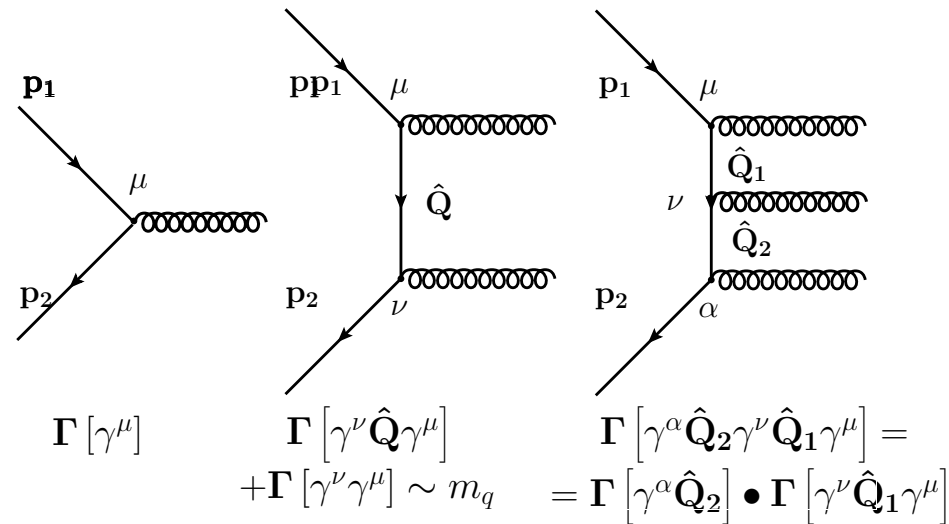


Figure 5: “Building” blocks for $q(p_1, \lambda_1) + \bar{q}(p_2, \lambda_2) \rightarrow g(k_1, \sigma_1) + g(k_2, \sigma_2) + g(k_3, \sigma_3)$

New block (in comparison to $q\bar{q} \rightarrow gg$) for this reaction with five γ -matrices can be evaluated with the help of universal block $\widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\mathcal{S}^n)$ (51)

Hence we have no new calculation structures of fermion part of matrix elements for $q\bar{q} \rightarrow gg \dots g$!!!

As an example of our calculation scheme let us consider matrix element $\mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_1, \sigma_2, \sigma_3}$ that is shown in Fig. 6. We use shorthand notation for D -function with $J = 1$:

$$D_{\lambda, \sigma}^{*1}(\theta_k, \varphi_k, -\varphi_k) \equiv \hat{k}[\lambda, \sigma] \quad (61)$$

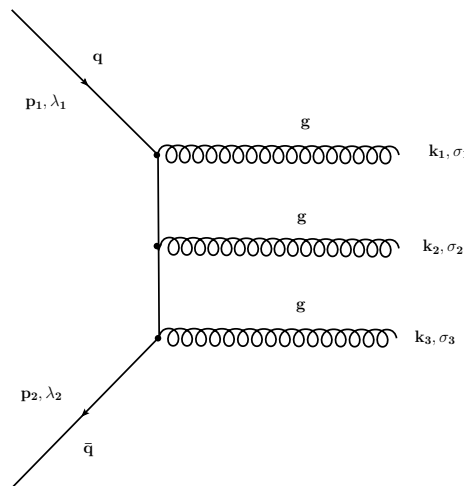


Figure 6:

Non-color part of this matrix element can be rewritten in terms of the block $\widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\mathcal{S}^n)$

$$\begin{aligned}
 \mathcal{M}_{\lambda_2, \lambda_1}^{\sigma_1, \sigma_2, \sigma_3} &\sim \widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\not{k}_3 \not{\epsilon}_{\sigma_3}(k_3) \not{\epsilon}_{\sigma_2}(k_2) \not{k}_1 \not{\epsilon}_{\sigma_1}(k_1)) + \\
 &+ 2(\epsilon_{\sigma_3}(k_3) \cdot p_2) \widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\not{\epsilon}_{\sigma_2}(k_2) \not{k}_1 \not{\epsilon}_{\sigma_1}(k_1)) + \\
 &+ 2(\epsilon_{\sigma_1}(k_1) \cdot p_1) \widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\not{k}_3 \not{\epsilon}_{\sigma_3}(k_3) \not{\epsilon}_{\sigma_2}(k_2)) - \\
 &- 4(\epsilon_{\sigma_3}(k_3) \cdot p_2)(\epsilon_{\sigma_1}(k_1) \cdot p_1) \widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\not{\epsilon}_{\sigma_2}(k_2)) .
 \end{aligned} \tag{62}$$

$$\begin{aligned}
 \widetilde{\mathcal{M}}_{\lambda_2, \lambda_1}(\not{k}_3 \not{\epsilon}_{\sigma_3}(k_3) \not{\epsilon}_{\sigma_2}(k_2) \not{k}_1 \not{\epsilon}_{\sigma_1}(k_1)) &= -\delta_{\sigma_1, -\sigma_3} k_1^{(0)} k_3^{(0)} \sigma_1 \sigma_2 \sigma_3 \sqrt{\frac{s}{2}} \times \\
 &\times \sqrt{(1 - \lambda_1 \sigma_1 \beta_q)(\lambda_2 \sigma_1 \beta_q + 1)} \sum_{A, B = -1}^1 A B \sqrt{AB + 3} \sqrt{(3 - A \lambda_2 \sigma_1)(B \lambda_1 \sigma_1 + 3)} \\
 &\hat{k}_2[-1/2\sigma_1(A + B), \sigma_2] \hat{k}_3[-1/2\sigma_3(A - \lambda_2 \sigma_1), \sigma_3] \hat{k}_1[1/2(B\sigma_1 + \lambda_1), \sigma_1] ,
 \end{aligned} \tag{63}$$

$$\begin{aligned}
 \widetilde{\mathcal{M}}_{\lambda_2, \lambda_1} (\not{\epsilon}_{\sigma_2} (k_2) \not{k}_1 \not{\epsilon}_{\sigma_1} (k_1)) &= -\frac{\sqrt{s}}{2} k_1^{(0)} \sigma_1 \sigma_2 \sqrt{1 - \lambda_1 \sigma_1 \beta_q} \sqrt{\lambda_2 \sigma_1 \beta_q + 1} \times \\
 &\times \sum_{A=-1}^1 A \sqrt{A \lambda_1 \sigma_1 + 3} \sqrt{A \lambda_2 \sigma_1 + 3} \times \\
 &\times \hat{k}_1 [1/2 (A \sigma_1 + \lambda_1), \sigma_1] \hat{k}_2 [-1/2 (A \sigma_1 + \lambda_2), \sigma_2] , \tag{64}
 \end{aligned}$$

$$\begin{aligned}
 \widetilde{\mathcal{M}}_{\lambda_2, \lambda_1} (\not{k}_3 \not{\epsilon}_{\sigma_3} (k_3) \not{\epsilon}_{\sigma_2} (k_2)) &= \frac{\sqrt{s}}{2} k_3^{(0)} \sigma_2 \sigma_3 \sqrt{(\lambda_1 \sigma_3 \beta_q + 1) (1 - \lambda_2 \sigma_3 \beta_q)} \times \\
 &\times \sum_{A=-1}^1 A \sqrt{A \lambda_1 \sigma_3 + 3} \sqrt{A \lambda_2 \sigma_3 + 3} \times \\
 &\times \hat{k}_2 [1/2 (A \sigma_3 + \lambda_1), \sigma_2] \hat{k}_3 [-1/2 (A \sigma_3 + \lambda_2), \sigma_3] \tag{65}
 \end{aligned}$$

$$\widetilde{\mathcal{M}}_{\lambda_2, \lambda_1} (\not{\epsilon}_{\sigma_2} (k_2)) = -\frac{\sigma_2}{\sqrt{2}} \sqrt{3 - \lambda_1 \lambda_2} (2m_q \delta_{\lambda_2, \lambda_1} + \sqrt{s} \delta_{\lambda_2, -\lambda_1}) \hat{k}_2 [1/2 (\lambda_1 - \lambda_2), \sigma_2] \tag{66}$$

Summary

We have formulated a new effective method to calculate Feynman amplitudes for various processes with fermions. **In our approach (MBS):**

1. We don't use an explicit form of Dirac spinors and γ -matrices (as well as basis spinors)
2. We don't use calculation of traces
3. Unlike different variants of spinor technique, in this method we don't use neither Chisholm identities, nor the presentation of the contraction \not{p} with four vector p and polarization vector of bosons through Dirac spinors
4. Unlike Weyl-van-der-Waerden spinor technique ([Giele87](#),[Dittmaier99](#)), in this method we don't use special Feynman rules for matrix elements calculation
5. Matrix element $\mathcal{M}_{\lambda_p, \lambda_k}^{\sigma_i}(p, s_p, k, s_k ; Q)$ is calculated for all values of $\lambda_p, \lambda_k, \sigma_i (i, 1, 2, 3 \dots)$ simultaneously.

The **MBS** enables us to calculate blocks of Feynman diagrams (current-like constructions and even more complicated structures) and then use them as universal functions during the process of calculation.

Thanks for your attention !!!

Danke sehr !!!

Merci Beaucoup !!!

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Viktor Andreev, "Scattering QCD amplitudes with massive fermions using recursive relations"